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PENETRATION OF A RIGID CONE INTO A PLASTIC ORTHOTROPIC HALF SPACE

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UDC 539.374

A study is made of the rigid-plastic flow of a plastic orthotropic material as a rigid rough cone penetrates a half space at a constant speed. The material of the half space is assumed to be incompressible, ideally rigid-plastic, and subject to the Mises–Hill relations [1] for a plastic orthotropic body. We assume that the principal axes of anisotropy coincide with the axes of a spherical coordinate system whose center is the vertex of the cone. An analogous problem for an isotropic material was studied in [2]; penetration of a rigid wedge into an anisotropic half space was considered in [3]; and the imbedding of a rigid stamp into an anisotropic plastic medium was investigated in [4]. A study of the penetration of a thin solid body into a transversally isotropic medium was given in [5]. In [6] a study was made of the penetration of a rigid cylindrical body into a plastic anisotropic pipe.

In the present paper we determine the pressure force during penetration of a rigid cone into a plastic orthotropic half space; we find the zone of distribution of plastic deformations and the form of the free surface of the displaced portion of half space material. A numerical example is presented showing the essential effect of anisotropy on the plastic zone distribution.

1. Assume that a rigid cone penetrates into a half space. We assume that the plastic region that is formed around the rigid cone of angle $\theta = \alpha$ is bounded by a conical surface with angle $\theta = \beta$; the location of this surface is to be determined in the course of solving our problem (Fig. 1). We assume that the region of plastic flow is bounded by a surface $r = R(\theta)$, free from external loads, whose shape is also to be determined. In this region properties of the material are assumed to be plastic orthotropic, being a consequence of plastic deformation of the material (deformation anisotropy). On the contacting conical surface there arises a tangential stress whose value depends mainly on the roughness of this surface.

Since there is no rotation of the rigid cone about its axis or the lateral area of the cone is ideally smooth in the peripheral direction, the annular component of the rate of displacement is equal to zero, whence $\gamma_{r\varphi} = \gamma_{\theta\varphi} = 0$, $\tau_{r\varphi} = \tau_{\theta\varphi} = 0$.

The differential equations of equilibrium in the spherical coordinate system then has the following form for our problem:

$$\begin{gathered}\frac{\partial\sigma_r}{\partial r} + \frac{1}{r}\frac{\partial\tau_{r\theta}}{\partial\theta} + \frac{1}{r}(2\sigma_r - \sigma_\theta - \sigma_\varphi + \tau_{r\theta}\operatorname{ctg}\theta) = 0,\\ \frac{\partial\tau_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial\sigma_\theta}{\partial\theta} + \frac{1}{r}[(\sigma_\theta - \sigma_\varphi)\operatorname{ctg}\theta + 3\tau_{r\theta}] = 0.\end{gathered} \tag{1.1}$$

Relations between the components of the deformation rate tensor, displacement rates, and stresses are:

$$\varepsilon_r = \frac{\partial u}{\partial r} = \Omega[H_0(\sigma_r - \sigma_\theta) + G_0(\sigma_r - \sigma_\varphi)],$$
$$\varepsilon_\theta = \frac{u}{r} + \frac{1}{r}\frac{\partial v}{\partial\theta} = \Omega[F_0(\sigma_\theta - \sigma_\varphi) + H_0(\sigma_\theta - \sigma_r)],$$

Erevan. Translated from Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 5, pp. 159-163, September-October, 1991. Original article submitted September 29, 1989; revision submitted April 25, 1990.

 0021-8944/91/3205-0808$12.50

$$\varepsilon_\varphi = \frac{u}{r} + \frac{v}{r}\operatorname{ctg}\theta = \Omega\,[G_0(\sigma_\varphi - \sigma_r) + F_0(\sigma_\varphi - \sigma_\theta)],$$

$$2\gamma_{r\theta} = \frac{\partial v}{\partial r} + \frac{1}{r}\frac{\partial u}{\partial \theta} - \frac{v}{r} = \Omega N_0 \tau_{r\theta}. \tag{1.2}$$

The Mises–Hill yield condition [1] may be written as

$$F_0(\sigma_\theta - \sigma_\varphi)^2 + G_0(\sigma_\varphi - \sigma_r)^2 + H_0(\sigma_r - \sigma_\theta)^2 + N_0\tau_{r\theta}^2 = 1. \tag{1.3}$$

Starting from relations (1.2), (1.3), we write the stress components in the form

$$\sigma_r = \sigma_\theta + (F\varepsilon_r - G\varepsilon_\theta)/\Omega,\ \sigma_\varphi = \sigma_\theta - [H\varepsilon_r + (G+H)\varepsilon_\theta]/\Omega,$$
$$\tau_{r\theta} = 2N\gamma_{r\theta}/\Omega, \tag{1.4}$$

where

$$\Omega = \left[(F+H)\,\varepsilon_r^2 + 2H\varepsilon_r\varepsilon_\theta + (G+H)\,\varepsilon_\theta^2 + 4N\gamma_{r\theta}^2\right]^{1/2},$$

$$F = F_0/\Delta,\ G = G_0/\Delta,\ H = H_0/\Delta,\ N = 1/N_0,\ \Delta = F_0G_0 + G_0H_0 + H_0F_0.$$

On the contact surface between the cone and the medium we specify the conditions

$$\tau_{r\theta} = m,\ v = V_0\sin\alpha\ (\theta = \alpha). \tag{1.5}$$

Here V_0 is the given rate of penetration of the cone; m is a positive constant whose value is assumed to be given and depends on the nature and degree of roughness of the conical surface in the radial direction.

On the boundary surface $\theta = \beta$ of the plastic zone we assume that the normal rate of displacement is continuous and that the tangential rate is discontinuous. We then put [7]

$$v = 0,\ \gamma_{r\theta} \to -\infty\ (\theta = \beta). \tag{1.6}$$

Next, we have a conservation of mass condition: the volume of the portion of the cone that has penetrated must equal the volume of material displaced.

2. In the plastic region the stress components and displacement rates may be expressed in terms of the unknown function $f(\theta)$:

$$\sigma_r = \sigma_\theta + \frac{G}{\sqrt{N(G+H)}}\sqrt{N-\tau^2},\quad \sigma_\varphi = \sigma_\theta + \sqrt{\frac{G+H}{N}}\sqrt{N-\tau^2},\ \tau_{r\theta} = \tau,$$
$$u = -f'\sin\theta - 2f\cos\theta,\ v = 2f\sin\theta,\ w = 0. \tag{2.1}$$

From Eqs. (1.1) we than have the expression

$$\sigma_\theta = -p_1 + A\ln\frac{r}{R_1} - 3\int_\alpha^\theta \tau\,d\theta + \sqrt{\frac{G+H}{N}}\int_\alpha^\theta \sqrt{N-\tau^2}\operatorname{ctg}\theta\,d\theta \tag{2.2}$$

and the following equation for determining the function $\tau(\theta)$:

$$\tau' = -A - \tau\operatorname{ctg}\theta + \frac{H-G}{\sqrt{N(G+H)}}\sqrt{N-\tau^2} \tag{2.3}$$

[A and p_1 are arbitrary constants, $R_1 = R(\alpha)$].

For the solution of Eq. (2.3) we have the boundary condition

$$\tau(\alpha) = m,\ \tau(\beta) = -\sqrt{N}. \tag{2.4}$$

The second condition in relations (2.4) is obtained by considering the bounding conical surface $\theta = \beta$ between the plastic and rigid zones as a surface of slippage.

We obtain function $f(\theta)$ from the differential equation

$$f'' + \left(3\operatorname{ctg}\theta - \sqrt{\frac{G+H}{N}}\,\frac{\tau}{\sqrt{N-\tau^2}}\right)f' = 0 \tag{2.5}$$

with boundary conditions

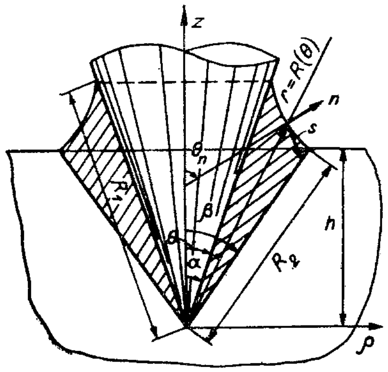

Fig. 1

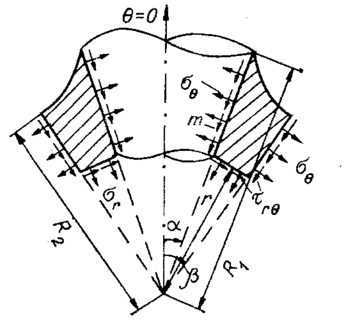

Fig. 2

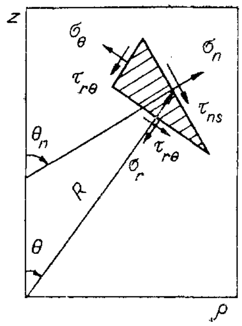

Fig. 3

$$f(\alpha)=V_0/2,\ f(\beta)=0, \tag{2.6}$$

as the solution of which we shall have

$$f(\theta)=\frac{V_0}{2}\frac{I(\theta)}{I(\alpha)},\quad I(\theta)=\int_\theta^\beta \exp\left(\sqrt{\frac{G+H}{N}}\int_\alpha^\theta \frac{\tau d\theta}{\sqrt{N-\tau^2}}\right)\frac{d\theta}{\sin^3\theta}. \tag{2.7}$$

Expressions for the displacement rates are then

$$u=\frac{V_0}{2I(\alpha)\sin^2\theta}\exp\left(\sqrt{\frac{G+H}{N}}\int_\alpha^\theta \frac{\tau d\theta}{\sqrt{N-\tau^2}}\right)-V_0\frac{I(\theta)}{I(\alpha)}\cos\theta,$$
$$v=V_0\sin\theta I(\theta)/I(\alpha),\ w=0. \tag{2.8}$$

3. Consider now the equilibrium of the cone-shaped tube occupying the plastic region $r\le\xi\le R(\theta)$, $\alpha\le\theta\le\beta$ (Fig. 2). Equating to zero the sum of the projections of all forces acting on the surfaces of the conceptually localized body in the direction of the $\theta=0$ axis, we obtain

$$\int_r^{R_1}[\sigma_\theta(r,\alpha)\sin\alpha-m\cos\alpha]\sin\alpha r dr-\int_r^{R_2}[\sigma_\theta(r,\beta)\sin\beta+$$
$$+\sqrt{N}\cos\beta]\sin\beta r dr-\int_\alpha^\beta[\sigma_r(r,\theta)\cos\theta-\tau_{r\theta}(\theta)\sin\theta]r^2\sin\theta d\theta=0. \tag{3.1}$$

Here $R_2=R(\beta)$. Substituting into Eq. (3.1) expressions for the stress components and carrying out the integration, we find

$$p_1=\frac{m\sin 2\alpha+\sqrt{N}\nu^2\sin 2\beta}{2(\nu^2\sin^2\beta-\sin^2\alpha)}+\frac{\nu^2\sin^2\beta\,(D+A\ln\nu)}{\nu^2\sin^2\beta-\sin^2\alpha}-\frac{A}{2},$$

$$\nu = \frac{R_2}{R_1}, \quad D = -3\int_\alpha^\beta \tau d\theta + \sqrt{\frac{G+H}{N}} \int_\alpha^\beta \sqrt{N-\tau^2}\,\mathrm{ctg}\,\theta d\theta.$$

Besides the functions $f(\theta)$, $\tau(\theta)$, and the parameter A, it remains to also determine function $R(\theta)$ and parameter β. From the condition for equilibrium of an element close to the free surface $r = R(\theta)$ in the meridian plane on an area with normal n (Fig. 3) we have [8]

$$\begin{aligned} \sigma_n &= \sigma_r(R,\theta)\cos^2\theta_* + \sigma_\theta(R,\theta)\sin^2\theta_* + \tau_{r\theta}(\theta)\sin 2\theta_*, \\ \tau_{ns} &= [\sigma_\theta(R,\theta) - \sigma_r(R,\theta)]\sin\theta_*\cos\theta_* + \tau_{r\theta}(\theta)\cos 2\theta_*, \end{aligned} \tag{3.2}$$

where $\theta_* = \theta_n - \theta$, and θ_n is the angle between the normal n and the positive z-direction. Let the parametric equation of the curve of intersection of the free surface with the meridian plane be given in the form $\rho = \rho(\theta)$, $z = z(\theta)$. Then

$$\sin\theta_n = -z' / \sqrt{\rho'^2 + z'^2}, \quad \cos\theta_n = \rho' / \sqrt{\rho'^2 + z'^2}. \tag{3.3}$$

Introducing the function

$$R(\theta) = h\exp[\chi(\theta)]/\cos\beta \tag{3.4}$$

($h = V_0 t = R_2\cos\beta$, h is a given depth, t is the time of penetration) and going over to a polar coordinate system, $\rho = R\sin\theta$, $z = R\cos\theta$, from relations (3.3) we obtain

$$\mathrm{tg}\,\theta_n = (\mathrm{tg}\,\theta - \chi')/(1 + \chi'\,\mathrm{tg}\,\theta). \tag{3.5}$$

As was done in [2], we form the expression $T = \sigma_n^2 + \tau_{ns}^2$ and determine R from conditions for a minimum of T with respect to θ_n. Upon differentiating and using relations (3.2), we find

$$\partial T/\partial\theta_n = 2(\sigma_r + \sigma_\theta)\tau_{ns}. \tag{3.6}$$

In the problem considered, we find, upon putting $\sigma_r + \sigma_\theta < 0$ and equating expression (3.6) to zero, that $\tau_{ns} = 0$.

Then

$$\theta_n = \theta + (1/2)\,\mathrm{arctg}\,[2\tau_{r\theta}/(\sigma_r - \sigma_\theta)]. \tag{3.7}$$

From these values of θ_n it follows from expression (3.6) that

$$\partial^2 T/\partial\theta^2 = -(\sigma_r + \sigma_\theta)\sqrt{(\sigma_r - \sigma_\theta)^2 + 4\tau_{r\theta}^2} > 0.$$

Consequently, function T attains a minimum value for θ_n so defined. On the surface $r = R(\theta)$, $\tau_{ns} = 0$, and the normal stress

$$\sigma_n(\theta) = \sigma_\theta(R,\theta) + (1/2)\left[\sigma_r - \sigma_\theta + \sqrt{(\sigma_r - \sigma_\theta)^2 + 4\tau_{r\theta}^2}\right]. \tag{3.8}$$

Substituting relation (3.7) into Eq. (3.5), we determine χ', and, then, taking into account the expressions (2.1) for the stress components, we obtain

$$\chi(\theta) = 2\int_\theta^\beta \frac{\tau d\theta}{\sqrt{\frac{G^2}{G+H} + \left[4 - \frac{G^2}{N(G+H)}\right]\tau^2 + \frac{G}{\sqrt{N(G+H)}}\sqrt{N-\tau^2}}}.$$

From function (3.4) we obtain the form of the displaced portion of the surface, namely

$$R(\theta) = \frac{h}{\cos\beta}\exp\left\{2\int_\theta^\beta \frac{\tau d\theta}{\sqrt{\frac{G^2}{G+H} + \left[4 - \frac{G^2}{N(G+H)}\right]\tau^2 + \frac{G}{\sqrt{N(G+H)}}\sqrt{N-\tau^2}}}\right\}. \tag{3.9}$$

Correspondingly, we have

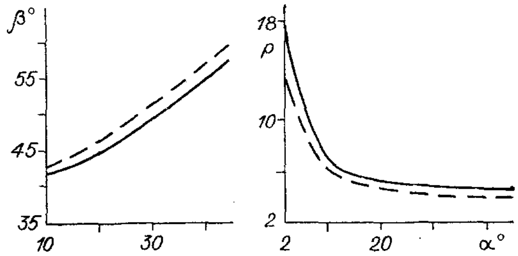

Fig. 4

$$\nu=\exp\left\{-2\int_{\alpha}^{\beta}\frac{\tau d\theta}{\sqrt{\frac{G^2}{G+H}+\left[4-\frac{G^2}{N(G+H)}\right]\tau^2+\frac{G}{\sqrt{N(G+H)}}\sqrt{N-\tau^2}}}\right\}. \tag{3.10}$$

The conservation of mass requirement yields

$$\int_{h}^{R_1\cos\alpha}\rho^2\,dz=\frac{1}{3}R_1^3\sin^2\alpha\cos\alpha.$$

Going over to polar coordinates, we obtain

$$\int_{\alpha}^{\beta}\exp(3\chi)\sin^3\theta\,(1-\chi'\operatorname{ctg}\theta)\,d\theta=\frac{1}{3\nu^3}\sin^3\alpha\cos\alpha.$$

Eliminating χ' with the aid of an integration by parts, we arrive at the equation

$$\frac{1}{2}\sin^2\beta\cos\beta=\int_{\alpha}^{\beta}\exp\left\{6\int_{\theta}^{\beta}\frac{\tau d\theta}{\sqrt{\frac{G^2}{G+H}+\left[4-\frac{G^2}{N(G+H)}\right]\tau^2+\frac{G}{\sqrt{N(G+H)}}\sqrt{N-\tau^2}}}\right\}\sin\theta\,d\theta, \tag{3.11}$$

which, together with relations (2.3) and (2.4), determines the function τ and parameters β and A.

The pressure force of the cone on the medium is

$$P=-2\pi\sin\alpha\int_{0}^{R_1}[\sigma_\theta(r,\alpha)\sin\alpha-m\cos\alpha]\,rdr.$$

We refer to the conditional stress, impressed on the base of the penetrated portion of the cone, i.e., $p = P/(\pi R_1^2\sin^2\alpha)$, as the specific pressure. After evaluating the integral, we have

$$p=\frac{m\sin 2\alpha+\sqrt{N}\nu^2\sin 2\beta}{2(\nu^2\sin^2\beta-\sin^2\alpha)}+\frac{\nu^2\sin^2\beta\,(D+A\ln\nu)}{\nu^2\sin^2\beta-\sin^2\alpha}+m\operatorname{ctg}\alpha.$$

According to the solution obtained, the surface $r = R(\theta)$ is "loaded" with the normally distributed forces $\sigma_n(\theta)$. It follows from relations (3.8) and (3.9) that

$$\sigma_n=-p_1+\frac{1}{2}\left\{\sqrt{\frac{G^2}{G+H}+\left[4-\frac{G^2}{N(G+H)}\right]\tau^2}+\frac{G}{\sqrt{N(G+H)}}\sqrt{N-\tau^2}\right\}-$$
$$-3\int_{\alpha}^{\theta}\tau\,d\theta-2A\int_{\alpha}^{\theta}\frac{\tau d\theta}{\sqrt{\frac{G^2}{G+H}+\left[4-\frac{G^2}{N(G+H)}\right]\tau^2+\frac{G}{\sqrt{N(G+H)}}\sqrt{N-\tau^2}}}+$$

$$+\sqrt{\frac{G+H}{N}}\int_{\alpha}^{\theta}\sqrt{N-\tau^2}\operatorname{ctg}\theta d\theta.$$

We assume that the effect of these loads on the stressed state of the plastic region far from the free surface is nonexistent. The relationship between the pressure force and the penetration depth is given by

$$P=\frac{\pi p}{\nu^2}\frac{\sin^2\alpha}{\cos^2\beta}h^2.$$

The problem (2.3), (2.4), (3.11) was solved numerically using the following simple algorithm. For the values selected for parameter β we solved the two-point boundary value problem (2.3), (2.4). Next, the condition (3.11) was verified. Condition (3.11) was satisfied by varying β. Finally, the quantities τ, β, and A were determined. Figure 4 shows, based on the numerical calculations, graphs of $\beta(\alpha)$ and $p(\alpha)$ for isotropic (dashed curves) and anisotropic (solid curves) materials for anisotropy parameters $G/N = 6$, $H/N = 2.5$ for for $m = 0.5$. In the case of an isotropic material $G/N = H/N = 2$. The graphs display the essential influence of anisotropy on the distribution of the plastic zone and the specific pressure of the cone on the medium.

A numerical study was also made of bounds for the variation of α. As $\alpha \to \pi/2$, $\beta \geq \pi/2$, which is inadmissible in accordance with the statement of the problem since $R_2 \to \infty$. This also depends on the anisotropy parameters and the value of m. With $\alpha < 80°$, and for all possible values of anisotropy parameters considered in our numerical studies, no such phenomenon was observed, i.e., the solution obtained for these α is completely admissible.

The author wishes to thank M. A. Zadoyan for his interest in the present paper.

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